

RIGID CHARACTERIZATIONS OF PSEUDOCONVEX DOMAINS

NIKOLAI NIKOLOV AND PASCAL J. THOMAS

ABSTRACT. We prove that an open set D in \mathbb{C}^n is pseudoconvex if and only if for any $z \in D$ the largest balanced domain centered at z and contained in D is pseudoconvex, and consider analogues of that characterization in the linearly convex case.

1. INTRODUCTION

Geometric convexity of a domain is characterized by its intersection with real lines, and invariant under real affine maps. Pseudoconvexity is a generalization of that notion that was designed, among other things, to be invariant under all biholomorphic maps, and can be characterized by the behavior of analytic disks (Kontinuitätsatz). Linear convexity and \mathbb{C} -convexity are intermediate notions that bring into play (respectively) complex hyperplanes and complex lines, and are invariant under complex affine maps.

In this paper, we exploit the parallels between all those notions, and highlight the similarities and differences, and the crucial role played by smoothness of the domains being considered.

2. BALANCED INDICATORS

Let D be an open set in \mathbb{C}^n , $z \in D$ and $X \in \mathbb{C}^n$. We say that a domain is *balanced, centered at a* if for any $z \in D$, $\zeta \in \mathbb{C}$ with $|\zeta| \leq 1$, then $a + \zeta(z - a) \in D$.

Denote by $d_D(z, X)$ the distance from z to ∂D in the complex direction X (possibly $d_D(z, X) = \infty$):

$$d_D(z, X) = \sup\{r > 0 : z + \lambda X \in D \text{ if } |\lambda| < r\}.$$

2000 *Mathematics Subject Classification.* 32F17.

Key words and phrases. pseudoconvex domain, (weakly) linearly convex domain, convex domain.

This paper was written during the stay of the first-named author at the Paul Sabatier University, Toulouse (October–November 2010) supported by a CNRS–BAS programme “Convention d’échanges” No 23811. He would like to thank Peter Pflug for helpful discussions about Proposition 14.

Recall that if $-\log d_D(\cdot, X)$ is a plurisubharmonic function for any $X \in \mathbb{C}^n$, then D is pseudoconvex, and vice versa.

Closely related to this is the largest balanced domain centered at z and contained in D , i.e. $B_{D,z} = z + I_{D,z}$, where $I_{D,z}$ is the balanced indicatrix of D at z :

$$I_{D,z} = \{X \in \mathbb{C}^n : z + \lambda X \in D \text{ if } |\lambda| \leq 1\}.$$

Finally, consider the global version of this, the Hartogs-like domain

$$H_D = \{(z, w) \in D \times \mathbb{C}^n : w \in I_{D,z}\}.$$

If D is pseudoconvex, then $-\log d_D$ is a plurisubharmonic function on $D \times \mathbb{C}^n$ (cf. [6, Proposition 2.2.21])¹, thus H_D is pseudoconvex.

2.1. Pseudoconvexity. The main purpose of this note is to characterize the pseudoconvexity of an open set D in \mathbb{C}^n in terms of pseudoconvexity of $B_{D,z}$, $z \in D$, i.e. in terms of pseudoconvexity in the "vertical" directions of H_D .

Theorem 1. *Let D be a proper open set of \mathbb{C}^n . Then the following properties are equivalent:*

- (1) D is pseudoconvex.
- (2) H_D is pseudoconvex.
- (3) $B_{D,z}$ is pseudoconvex, for any $z \in D$.

We have already seen that (1) implies (2), and (2) implies (3) is trivial (slice by the sets $\{z\} \times \mathbb{C}^n$, for $z \in D$). The remaining implication is implied by the following.

Proposition 2. *Let D be a proper open set of \mathbb{C}^n and let U be a neighborhood of ∂D . If $I_{D,a}$ is a pseudoconvex domain for any $a \in D \cap U$, then D is itself pseudoconvex.*

To prove Proposition 2 (and other propositions below), we shall use [1, Theorem 4.1.25]², namely

Proposition 3. *If an open set D in \mathbb{C}^n is not pseudoconvex, then there is a point $a \in \partial D$, say the origin, and a real-valued quadratic polynomial q such that $q(a) = 0$, $\partial q(a) \neq 0$,*

$$\sum_{j,k=1}^n \frac{\partial^2 q}{\partial z_j \partial \bar{z}_k}(a) X_j \overline{X_k} < 0$$

¹The authors thank P. Pflug for pointing out this fact.

²The first inequality on p. 242 in the proof must contain an obvious extra term. Otherwise, it is not true in general; for example, take the domain in \mathbb{C}^2 given by $\operatorname{Re} z < (\operatorname{Re} w)^2$. But the end result does hold.

for some vector $X \in \mathbb{C}^n$ with $\langle \partial q(a), X \rangle = 0$, and D contains the set $\{q < 0\}$ near a .

Therefore, after an affine change of coordinates, we may assume $0 \in \partial D$ and, near this point, D contains the set

$$\{z \in \mathbb{C}^n : 0 > \operatorname{Re} z_1 + (\operatorname{Im} z_1)^2 + |z_2|^2 + \cdots + |z_{n-1}|^2 + c(\operatorname{Im} z_n)^2 - (\operatorname{Re} z_n)^2\},$$

where $c < 1$.

Proof of Proposition 2. Assume that D is not pseudoconvex. By Proposition 3, we may suppose that

$$D \supset E = \{(z, w) \in \mathbb{D}_\varepsilon^2 : \rho(z, w) < 0\},$$

where $\rho(z, w) = \operatorname{Re} z + (\operatorname{Im} z)^2 - (\operatorname{Re} w)^2 + c(\operatorname{Im} w)^2$ and $c < 1$ (\mathbb{D}_ε^2 is the bidisc with center 0 and radius $\varepsilon > 0$).

For $\delta > 0$ and $X \in \mathbb{C}^2$, let $z_\delta = (-\delta, 0)$ and $r_\delta(X) = d_E(z_\delta, X)$. We write, for $\eta \in \mathbb{C}$, $X_\eta = (\delta, \eta)$.

Lemma 4. *For any small $\delta > 0$ and $\delta \geq s \geq 3(1 - c)^{-1/2}\delta^{3/2}$,*

$$\int_0^{2\pi} \frac{1}{r_\delta(X_{se^{i\theta}})} \frac{d\theta}{2\pi} < 1.$$

Assuming Lemma 4, set $\mathbb{C}^n \ni \tilde{z}_\delta = (-\delta, 0, \dots, 0)$ and $\mathbb{C}^n \ni \tilde{X}_\eta = (\delta, \eta, 0, \dots, 0)$. Since $r_\delta \leq d_{D, \tilde{z}_\delta}$ and $r_\delta(X_0) = d_D(\tilde{z}_\delta, X_0) = 1$ for δ small enough, it follows that $h_\delta = 1/d_D(\tilde{z}_\delta, \cdot)$ is not a plurisubharmonic function, which implies that the balanced domain I_{D, \tilde{z}_δ} (with Minkowski function h_δ) is not a pseudoconvex domain (cf. [6, Proposition 2.2.22 (a)]). This contradiction proves Proposition 2. \square

Lemma 4 will be proved at the end of this section.

2.2. Linear convexity. It is interesting to note that a similar statement holds for linear convexity. Recall that (cf. [1]) a open set D in \mathbb{C}^n is called weakly linearly convex (resp. linearly convex) if for any $a \in \partial D$ (resp. $a \in \mathbb{C}^n \setminus D$) there exists a complex hyperplane T_a through a which does not intersect D (such a set is necessarily pseudoconvex). We call T_a a supporting complex hyperplane. A domain D in \mathbb{C}^n is said to be \mathbb{C} -convex if any nonempty intersection of D with a complex line is connected and simply connected. All three notions coincide for C^1 -smooth open sets.

Note that an open balanced set is weakly linearly convex if and only if it is convex. It is also known that if D is weakly linearly convex, then $B_{D, z}$ is a convex domain for any $z \in D$ (i.e. the Minkowski function $1/d_D(z, \cdot)$ of $I_{D, z}$ is convex).

Theorem 5. *Consider the following three properties:*

- (1) *D is weakly linearly convex (resp. linearly convex).*
- (2) *H_D is weakly linearly convex (resp. linearly convex).*
- (3) *$B_{D,z}$ is (weakly linearly) convex, for any $z \in D$.*

Then (1) and (2) are equivalent, and imply (3). If D is a $C^{1,1}$ -smooth bounded domain, then (3) implies (1).

The last statement follows from [7]. Note that in this case, the domain D is in fact \mathbb{C} -convex. The domain H_D , however, does not share the smoothness of D , and may fail to be \mathbb{C} -convex.

Example 6. *If $D = \{z \in \mathbb{C} : |z - 1| < 2 \text{ or } |z + 1| < 2\}$, then H_D is not \mathbb{C} -convex.*

Proof. The set $H_D \cap (\mathbb{C} \times \{\sqrt{3}\})$ is not connected. \square

Proof of Theorem 5. Since $D = (\mathbb{C}^n \times \{0\}) \cap H$, (2) implies (1). To prove the converse, may assume that $D \neq \mathbb{C}^n$. Let D be weakly linearly convex (resp. linearly convex) and $(a, b) \in \partial D$ (resp. $(a, b) \in \mathbb{C}^n \setminus D$). It follows that $c = a + \lambda_0 b \in \partial D$ (resp. $c \in \mathbb{C}^n \setminus D$) for some $\lambda_0 \in \mathbb{C}$ with $|\lambda_0| \leq 1$. There exists a supporting complex hyperplane for D at c , say $T_c = \{z \in \mathbb{C}^n : L(z) = 0\}$, where $L : \mathbb{C}^n \rightarrow \mathbb{C}$ is an affine map. Then $T_{a,b} = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^n : L(z + \lambda_0 w) = 0\}$ is a supporting complex hyperplane for H_D at (a, b) . \square

If we turn to the third, and more usual notion of convexity, it is clear that a domain D in \mathbb{R}^n is convex if and only if H_D is convex in $\mathbb{R}^n \times \mathbb{R}^n$.

2.3. Proof of Lemma 4. Set $\zeta = re^{i\alpha}$, $-\pi < \alpha \leq \pi$, and $C_1 = 3\sqrt{\delta}$. We first estimate $\rho(z_\delta + \zeta X_{se^{i\theta}})$ when $|\alpha| \geq C_1$:

$$\begin{aligned} \rho(z_\delta + \zeta X_{se^{i\theta}}) &\leq \delta(-1 + r \cos \alpha + \delta r^2 \sin^2 \alpha) + cr^2 s^2 \\ &\leq \delta(-1 + r \cos C_1) + 2r^2 \delta^2 \leq \delta(-1 + r(\cos C_1 + 2\delta)). \end{aligned}$$

For any small δ , $\cos C_1 \leq 1 - C_1^2/3 = 1 - 3\delta$, so if $r < (1 - \delta)^{-1}$, then $\rho(z_\delta + \zeta X_{se^{i\theta}}) < 0$.

Now we estimate $\rho(z_\delta + \zeta X_{se^{i\theta}})$ when $|\alpha| \leq C_1$. Notice that

$$\rho(z_\delta + \zeta X_{se^{i\theta}}) = \delta(-1 + r \cos \alpha + \delta r^2 \sin^2 \alpha) + r^2 s^2 ((1 + c) \sin^2(\alpha + \theta) - 1).$$

It is easy to check that

$$\sin^2(\alpha + \theta) \leq \sin^2 \theta + \sin |\alpha| \leq \sin^2 \theta + C_1,$$

so

$$\begin{aligned}\rho(z_\delta + \zeta X_{se^{i\theta}}) &\leq \delta(-1 + r + \delta r^2 C_1^2) + r^2 s^2 ((1+c)\sin^2 \theta + (1+c)C_1 - 1) \\ &= \delta(-1 + r + r^2 A),\end{aligned}$$

where $A = A(\theta) := \delta C_1^2 + \frac{s^2}{\delta} ((1+c)\sin^2 \theta + (1+c)C_1 - 1)$. Notice that $-1 \leq -\delta < A$ for $s \leq \delta \leq 1$.

Suppose that $1/r > 1 + A$, then

$$\frac{1}{\delta} \rho(z_\delta + \zeta X_{e^{i\theta}}) < -\frac{A^2}{(1+A)^2} \leq 0.$$

Putting together both estimates, for δ small enough,

$$r_\delta(X_{se^{i\theta}}) > \min\left(\frac{1}{1-\delta}, \frac{1}{1+A}\right) = \frac{1}{1+A}.$$

Therefore

$$\begin{aligned}\int_0^{2\pi} \frac{1}{r_\delta(X_{se^{i\theta}})} \frac{d\theta}{2\pi} &< \int_0^{2\pi} (1+A(\theta)) \frac{d\theta}{2\pi} \\ &= 1 + \delta C_1^2 + \frac{s^2}{\delta} \left((1+c)\frac{1}{2} + (1+c)C_1 - 1 \right) \leq 1 + 3\delta^2 - \frac{s^2(1-c)}{3\delta} \leq 1\end{aligned}$$

for any small δ . \square

3. DEFINING FUNCTIONS

3.1. Convexity. We point out that the proof that the convexity of $B_{D,z}$ implies linear convexity for $\mathcal{C}^{1,1}$ domains [7, Proposition 1 & introduction] is based on the following which can be easily deduced from [2]. Let s_D stand for the signed distance to ∂D .

Proposition 7. *If D is a $C^{1,1}$ -smooth bounded domain in \mathbb{C}^n and*

$$\liminf_{T_a^{\mathbb{C}} \ni z \rightarrow a} \frac{s_D(z)}{|z-a|^2} \geq 0^3$$

for $a \in \partial D$ almost everywhere, then D is linearly convex.

Proposition 7 has an obvious convex analog.

Proposition 8. *A proper domain D in \mathbb{R}^n is convex if and only if for any $a \in \partial D$ there exists a (real) hyperplane S_a through a such that*

$$\liminf_{S_a \ni x \rightarrow a} \frac{s_D(x)}{|x-a|^2} \geq 0.$$

³When ∂D is twice differentiable at a , this limit is equal to the minimal eigenvalue of $2\text{Hess}_{s_D}(a)|_{T_a^{\mathbb{C}}}$.

If D is convex, then obviously S_a is a (real) supporting hyperplane.

Proof. The necessity is clear by taking supporting hyperplanes.

Assume now that D is not convex. By [1, Theorem 2.1.27], one may find $a \in \partial D$ and a smooth domain $G \subset D$ such that $a \in \partial G$ and $2\text{Hess}_{s_G}(a)|_{T_a^{\mathbb{R}}}$ has an eigenvalue $\lambda < 0$. Since $s_D \leq s_G$, it follows that

$$\liminf_{T_a^{\mathbb{R}} \ni x \rightarrow a} \frac{s_D(x)}{|x - a|^2} \leq \lambda$$

and

$$\liminf_{t \rightarrow 0} \frac{s_D(a + tX)}{t^2} = -\infty, \quad a + X \notin T_a^{\mathbb{R}}$$

which implies the sufficient part. \square

Clearly, the relationship between a domain and its defining function is not symmetric, as convexity of one sublevel set (or indeed, of all of them) cannot imply convexity of the function: simply compose by a monotone increasing function from the real line to itself. Given a convex domain, the question arises of how to choose a convex defining function, and of how much choice one may have.

By [3, Proposition], a smooth bounded domain D is convex if and only if $-\log s_D$ is convex near ∂D . Thanks to [1, Theorem 2.1.27], this result can be easily generalized.

Proposition 9. *Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a nonconstant decreasing and convex function. Let U be a neighborhood of the boundary of a proper domain D in \mathbb{R}^n . Then D is convex if and only if $f \circ s_D$ is a convex function on $D \cap U$.*

In particular, if any of the defining functions given above is convex on a neighborhood of ∂D , then all the others are.

Proof. If D is convex, then s_D is concave and thus $g = f \circ s_D$ is convex on D .

To prove the converse, assume that g is convex on $D \cap U$ but D is not convex. By [1, Theorem 2.1.27] (see the proof of Proposition 8), we may find a segment $[a, b] \in D \cap U$ such that $s_D(m) < s_D(x)$ for any $x \in [a, b] \setminus \{m\}$, where $m = \frac{a+b}{2}$. On the other hand, it follows that f is strictly decreasing and then $g(a) + g(b) < 2g(m)$, a contradiction. \square

Note that it is necessary to require that the function f be decreasing and convex as the following example shows.

Example 10. *Let $D = \mathbb{R}^+ \times \mathbb{R}^+$ and let $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a nonconstant function such that $f \circ s_D$ is a convex function on D . Then f is decreasing and convex.*

Proof. Let $g = f \circ s_D$. Since $g(t, t) = f(t)$ for $t > 0$, it follows that f is convex. On the other hand, $2f(t) = 2g(t, t) \leq g(p, t) + g(2t - p, t) = f(p) + f(t)$, i.e. $f(t) \leq f(p)$ for $0 < p \leq t$. \square

3.2. Pseudoconvexity. The pseudoconvex analog of Proposition 9 is the following.

Proposition 11. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a nonconstant increasing and convex function. Let U be a neighborhood of the boundary of a proper domain D in \mathbb{C}^n . Then D is pseudoconvex if and only if $f \circ q_D$ is a plurisubharmonic function on $D \cap U$, where $q_D = -\log s_D$.*

Proof. If D is pseudoconvex, then q_D is plurisubharmonic and thus $g = f \circ s_D$ is plurisubharmonic on D .

To prove the converse, assume that g is plurisubharmonic on $D \cap U$ but D is not pseudoconvex. Note that there is $m \in \mathbb{R}$ such that $f(x)$ is strictly increasing for $x > m$. We may assume that $-\log s_D > m$ on $D \cap U$. Using Proposition 3, we may easily find a quadratic polynomial map $p \in \mathcal{O}(\mathbb{D}, D \cap U)$ (\mathbb{D} is the unit disc) such that $s_D(p(0)) < s_D(p(\zeta))$ for any $\zeta \in \mathbb{D}_*$ (strong Kontinuitätsatz). Then $g(p(0)) > g(p(\zeta))$ which contradicts to the maximum principle for the subharmonic function $g \circ p$. \square

Note that there is a smooth bounded pseudoconvex domain in \mathbb{C}^2 having no defining function which is plurisubharmonic on a two-sided neighborhood of the boundary. We do not know under which general conditions on f the plurisubharmonicity of $f \circ s_D$ is equivalent to the pseudoconvexity of D .

We also point out that the proofs of Propositions 8 and 11 imply a similar result to Propositions 7 and 8 in the pseudoconvex case:

Proposition 12. *If D is a proper open set in \mathbb{C}^n and for any $a \in \partial D$ there exists a complex hyperplane S_a through a such that*

$$\liminf_{S_a \ni z \rightarrow a} \frac{s_D(z) + s_D(a + J(z - a))}{|z - a|^2} \geq 0,$$

where J is the standard complex structure, then D is pseudoconvex.

The converse is also true if D is a C^2 -smooth open set. We do not know if the smoothness can be weakened.

4. SLICING

It is known that an open set D in \mathbb{C}^n ($n \geq 3$) is pseudoconvex if and only if any 2-dimensional slice of D is pseudoconvex [5] (see also [4]). Following the idea in [4], we would like to restrict the family of

slices that has to be used in order to detect pseudoconvexity, namely we would like to consider the family of complex planes passing through a common point $a \in \mathbb{C}^n$. As the next results show, it will be enough generically. Given D be a open non-pseudoconvex set in \mathbb{C}^n , call a *exceptional* with respect to D if for any 2-dimensional complex plane $P \ni a$, $P \cap D$ is pseudoconvex. The next proposition shows that the set of exceptional points has to be contained in a complex hyperplane.

Proposition 13. *Let D be a open non-pseudoconvex set in \mathbb{C}^n ($n \geq 3$). Let S be the union of all 2-dimensional complex planes with non-empty and non-pseudoconvex intersections with D , so the set of exceptional points is $\mathbb{C}^n \setminus S$. Then there exists a complex hyperplane T such that $\mathbb{C}^n \setminus S \subset T$.*

Proof. By Proposition 3, we may suppose that $0 \in \partial D$ and $D \supset G \cap \mathbb{D}^n(0, \varepsilon)$ for some $\varepsilon > 0$, where

$$G = \{z \in \mathbb{C}^n : 0 > r(z) = \operatorname{Re} z_1 + \|z\|^2 - c(\operatorname{Re} z_n)^2\}, \quad c > 2.$$

Choose a point $a \in \mathbb{C}^n$ which does not belong to the complex tangent hyperplane to ∂G at 0, i.e. with non-zero first coordinate. It is enough to show that if $L = \operatorname{span}(\vec{a}, \vec{e}_n)$, then $D' = D \cap L$ is not pseudoconvex. For this, note that since $\{z_1 = 0\} \cap L$ is a transverse intersection, $G \cap L$ is a smooth domain near 0 and since $\mathbb{C}e_n \subset \{z_1 = 0\} \cap L$, the Levi form of $r|_L$ has a negative eigenvalue at 0. Set $G' = G \cap L \cap \mathbb{D}^n(0, \varepsilon)$. Then there is a quadratic polynomial map $\varphi \in \mathcal{O}(\mathbb{D}, G') \subset \mathcal{O}(\mathbb{D}, D')$ with

$$\|\varphi(0)\| < \operatorname{dist}(\varphi(\zeta), \partial G') \leq \operatorname{dist}(\varphi(\zeta), \partial D'), \quad \zeta \in \mathbb{D}_*$$

(we have already used this argument in the proof of Proposition 11) which shows that D' is not pseudoconvex. \square

If D is C^2 -smooth, the set of exceptional points with respect to D has to be smaller (compare with Example 16(i)).

Proposition 14. *Let D and S be as in Proposition 13. If D is C^2 -smooth and non-pseudoconvex near some boundary point, then there exists a complex plane T of codimension 3 such that $\mathbb{C}^n \setminus S \subset T$.*

Proof. Note that the respective boundary point, say 0, satisfies the conclusions of Proposition 3. We have the same for any point $a \in \partial D$ near 0. Assume that $u, v \notin S$. Let r be a C^2 -smooth defining function for D near 0. By the proof of Proposition 13, the complex tangent hyperplane to ∂D at any such a contains u and v . Then it is easy to see that the derivative of r in direction $u - v$ vanishes at any such a . So, if $\mathbb{C}^n \setminus S$ does not lie in a complex plane of codimension 3, we may

assume that near 0, r depends only on two coordinates, z_1 and z_n , and so we can take (as in the proof above)

$$G = \{z \in \mathbb{C}^n : 0 > r_G(z) := \operatorname{Re} z_1 + (\operatorname{Im} z_1)^2 + |z_n|^2 - c(\operatorname{Re} z_n)^2\}, \quad c > 2.$$

We already know that for any exceptional point b , $b_1 = 0$. Suppose that $b_n \neq 0$. Consider the complex plane P through 0 spanned by b and e_1 . Then the complex tangent space to $P \cap D$ at 0 is $\mathbb{C}e_n$,

$$r_G(\zeta e_1 + \xi b) = r_G(\zeta_1 e_1 + \xi b_n e_n),$$

so that the intersection with that plane is not pseudoconvex at 0. Therefore the set of exceptional points is contained in $\{z_1 = z_n = 0\}$.

Let $D' = D \cap \{z_2 = \cdots = z_{n-1} = 0\}$. The complex tangent space to any point in ∂D near 0 must pass through an exceptional point, so the complex tangent line to any point in $\partial D'$ near 0 passes through 0. In particular, the same holds for the real tangent hyperplanes.

Taking a C^1 -smooth defining function of D' near 0 of the form $x_1 - \rho(u)$, where $u = (y_1, x_n, y_n)$, we get the Euler differential equation

$$\rho(\tilde{u}) = \frac{\partial \rho(\tilde{u})}{\partial y_1} \tilde{y}_1 + \frac{\partial \rho(\tilde{u})}{\partial x_n} \tilde{x}_n + \frac{\partial \rho(\tilde{u})}{\partial y_n} \tilde{y}_n.$$

Hence ρ is a homogeneous function of order 1 and the C^1 -smoothness near 0 implies that ρ is linear. It follows that $\partial D'$ is a hyperplane near 0 and so it is pseudoconvex there, which is a contradiction. \square

The following example shows that there can be an exceptional point even in the 3-dimensional case, when the boundary is smooth except one point.

Example 15. *There exists a an unbounded domain in \mathbb{C}^3 with real-analytic boundary except one point, the origin, which has exactly one exceptional point, namely the origin.*

Proof. Let $\Omega = \{|z_3|^2 < |z_1|^2 + |z_2|^2 < 4|z_3|^2\}$ and $\rho(z) = |z_3|^2 - |z_1|^2 - |z_2|^2$. For a point $0 \neq z^0 = (z_1^0, z_2^0, z_3^0)$ with $\rho(z^0) = 0$, the restriction of ρ to the horizontal complex line inside the complex tangent hyperplane is

$$\rho(z_1^0 - \lambda \bar{z}_2^0, z_2^0 + \lambda \bar{z}_1^0, z_3^0) = -|\lambda|^2(|z_1^0|^2 + |z_2^0|^2).$$

Using homogeneity, it is easy to check that the Levi form of ρ is semi-definite negative with one strictly negative eigenvalue (in particular, $\rho \equiv 0$ along the line through the origin and (z_1^0, z_2^0, z_3^0)).

Now let $P = \{\alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3 = 0\}$. If $\alpha_3 \neq 0$, since Ω is invariant under rotations in the (z_1, z_2) -plane, we may assume $P = \{z_3 = \alpha z_1\}$, where $\alpha \geq 0$. Set $D_\alpha = \{z \in \mathbb{C}^3 : |z_2|^2 < (4\alpha^2 - 1)|z_1|^2\}$ for $\alpha > 1/2$, and $G_\alpha = \{z \in \mathbb{C}^3 : (\alpha^2 - 1)|z_1|^2 < |z_2|^2\}$ for $\alpha > 1$. Note that both

domains are pseudoconvex. Then $\Omega \cap P = \emptyset$ if $\alpha \leq 1/2$, $\Omega \cap P = D_\alpha \cap P$ if $1/2 < \alpha < 1$, $\Omega \cap P = D_\alpha \cap P \cap (\mathbb{C} \times \mathbb{C}_* \times \mathbb{C})$ if $\alpha = 1$ and $\Omega \cap P = D_\alpha \cap G_\alpha \cap P$ if $\alpha > 1$; these intersections are pseudoconvex.

If $\alpha_3 = 0$, using a rotation again, we may assume $P = \{z_2 = 0\}$. Then $\Omega \cap P = \{|z_3|^2 < |z_1|^2 < 4|z_3|^2\}$ which is pseudoconvex.

So, the origin is an exceptional point.

On the other hand, it follows by the proof of Proposition 13 that an exceptional point belongs to the complex tangent hyperplane to ∂D at any z^0 as above. This implies that the origin is the only exceptional point. \square

In the 3-dimensional case we may have more than one exceptional point.

Example 16. Let $a \in \mathbb{C}^3$, G be a pseudoconvex set in \mathbb{C}^3 , and let l_1, l_2 be distinct complex lines in \mathbb{C}^3 that intersect G . Then:

- (i) any intersection of $G \setminus l_1$ with a 2-dimensional complex plane through a is pseudoconvex if and only if $a \in l_1 \setminus G$.
- (ii) any intersection of $G \setminus (l_1 \cup l_2)$ with a 2-dimensional complex plane through a is pseudoconvex if and only if $G \not\ni a = l_1 \cap l_2$.

Proof. (i) Let P be a 2-dimensional complex hyperplane through a . Let first $a \in l_1 \setminus G$. If $l_1 \not\subset P$, then $G_1 := (G \setminus l_1) \cap P = G \cap P =: G_P$ is pseudoconvex. Otherwise, G_1 is pseudoconvex as the intersection of the pseudoconvex sets G_P and $P \setminus l_1$.

Let now $a \notin l_1 \setminus G$. If $a \in G \cap l_1$ and P contains no l_1 , then $G_1 = G_P \setminus \{a\}$ is not pseudoconvex. Otherwise, $a \notin l_1$ and if P intersect l_1 at $b \in G$, then $G_1 = G_P \setminus \{b\}$ is not pseudoconvex.

(ii) The proof is similar to that of (i) and we skip it. \square

Using Proposition 7, similar arguments as in the proof of Proposition 13 implies that a is a point in C^2 -smooth domain D such that any non-empty intersection of D with a 2-dimensional complex plane through a is weakly linearly convex, then D is \mathbb{C} -convex.

The following example shows that we have no such phenomenon in general.

Example 17. Let

$$D = \{z \in \mathbb{C}^3 : |z| < \sqrt{2} \max\{|z_1|, |z_2|, |z_3|\}\}.$$

Then D is a union of three disjoint linearly convex domains and D has a non-empty linearly convex intersection with any complex plane through 0 (in particular, D is pseudoconvex and not weakly linearly convex).

Proof. Letting $D_j = \{z \in \mathbb{C}^3 : |z_j|^2 > \sum_{1 \leq k \leq 3, k \neq j} |z_k|^2\}$, we see that $D = \cup_{j=1}^3 D_j$. Clearly the D_j are pairwise disjoint, and obtained one from the other by unitary transformations (permutations of coordinates).

$\mathbb{C}^3 \setminus D_3$ is the union of the complex planes of the form $\{z_3 = \alpha_1 z_1 + \alpha_2 z_2\}$ for $|\alpha_1|^2 + |\alpha_2|^2 \leq 1\}$. So all D_j are linearly convex.

For any complex plane P , $D \cap P$ is a union of punctured complex lines through 0, so (as in the proof of Example 15), it is pseudoconvex if and only if it is smaller than $P \setminus \{0\}$. But $P \setminus \{0\}$ is connected, so if $P \setminus \{0\} \subset D$, then there exists a j such that $P \setminus \{0\} \subset D_j$. Since D_j is linearly convex, it must be pseudoconvex, so by Hartog's phenomenon, it would contain 0, a contradiction.

To finish the proof and show that D is not linearly convex, we will show that its complement contains no complex plane P . By permuting coordinates, we may assume that $P = \{z_3 = \alpha_1 z_1 + \alpha_2 z_2\}$, with $|\alpha_1| \leq 1$, $|\alpha_2| \leq 1$. If we suppose that one of those inequalities is strict, say $|\alpha_1| < 1$, then the points in P such that $z_2 = 0$ verify $|z_1|^2 > |\alpha_1 z_1|^2 = |z_3|^2 + |z_2|^2$ and $P \cap D_1 \neq \emptyset$. If $|\alpha_1| = |\alpha_2| = 1$, there are points $(z_1, z_2, z_3) \in P$ such that $|z_3| = |z_1| + |z_2|$ and $z_1 z_2 \neq 0$, so $|z_3|^2 = |z_1|^2 + |z_2|^2 + |z_1||z_2| > |z_1|^2 + |z_2|^2$, thus $P \cap D_3 \neq \emptyset$. \square

In spite of Example 17, one may also conjecture the following:

If D is an open set in \mathbb{C}^n such that any non-empty intersection with 2-dimensional complex plane is (weakly) linearly convex, then D is (weakly) linearly convex.

REFERENCES

- [1] L. Hörmander, *Notions of convexity*, Birkhäuser, Basel–Boston–Berlin, 1994.
- [2] L. Hörmander, *Weak linear convexity and a related notion of concavity*, Math. Scan. 102 (2008), 73–100.
- [3] A.-K. Herbig, J. D. McNeal, *Convex defining functions for convex domains*, J. Geom. Anal., DOI: 10.1007/s12220-010-9202-8.
- [4] R. Jacobson, *Pseudoconvexity is a two-dimensional phenomenon*, arXiv: 0907.1304.
- [5] S. Hitotunatu, *On some conjectures concerning pseudo-convex domains*, J. Math. Soc. Japan Volume 6 (1954), 177–195.
- [6] M. Jarnicki, P. Pflug, *Extension of holomorphic functions*, de Gruyter, Berlin, 2000.
- [7] N. Nikolov, P. J. Thomas, “Convex” characterization of linearly convex domains, arXiv:1009.5022.

INSTITUTE OF MATHEMATICS AND INFORMATICS, BULGARIAN ACADEMY OF
SCIENCES, 1113 SOFIA, BULGARIA

E-mail address: `nik@math.bas.bg`

UNIVERSITÉ DE TOULOUSE, UPS, INSA, UT1, UTM, INSTITUT DE MATHÉ-
MATIQUES DE TOULOUSE, F-31062 TOULOUSE, FRANCE

E-mail address: `pthomas@math.univ-toulouse.fr`